

Potpourri, 5

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Let $\{a_j\}_{j=1}^\infty$ be a sequence of nonnegative real numbers which is *submultiplicative* in the sense that

$$(1) \quad a_{j+l} \leq a_j a_l$$

for all positive integers j, l . Let us show that the limit

$$(2) \quad \lim_{l \rightarrow \infty} (a_l)^{1/l}$$

exists, and in fact is equal to the infimum of the a_j 's. If $a_j = 0$ for some j , then $a_l = 0$ for all $l \geq j$, and so we may as well assume that $a_j > 0$ for all j .

Notice first that

$$(3) \quad a_j \leq (a_1)^j$$

for all positive integers j , so that the sequence consisting of $(a_l)^{1/l}$ for all positive integers l is bounded. More generally, $a_{pk} \leq (a_k)^p$ for all positive integers k and p , so that $(a_n)^{1/n} \leq (a_k)^{1/k}$ when n is a multiple of k .

Fix a positive integer k , and let n be a positive integer such that $k \leq n$. We can write n as $pk + r$, where p, r are integers, $p \geq 1$, and $0 \leq r < k$. Thus

$$(4) \quad a_n \leq (a_k)^p (a_1)^r$$

by submultiplicativity.

We can rewrite this as

$$(5) \quad (a_n)^{1/n} \leq (a_k)^{1/(k+(r/p))} (a_1)^{r/n}.$$

These notes are connected to a “potpourri” class in the mathematics department at Rice University.

Using this one can check that $\limsup_{n \rightarrow \infty} (a_n)^{1/n}$ is less than or equal to $(a_k)^{1/k}$ for all positive integers k , from which it follows that $\{(a_l)^{1/l}\}_{l=1}^{\infty}$ converges to the infimum of the a_j 's, as desired.

Let us review some notions which will be used here frequently. If V is a vector space over the real or complex numbers, then a function $N(v)$ on V is called a *seminorm* if $N(v)$ is a nonnegative real number for all $v \in V$ which is equal to 0 when $v = 0$ and satisfies

$$(6) \quad N(\alpha v) = |\alpha| N(v)$$

for all real or complex numbers α , as appropriate, and all $v \in V$, and

$$(7) \quad N(v + w) \leq N(v) + N(w)$$

for all $v, w \in V$. If furthermore $N(v) > 0$ when v is a nonzero vector in V , then N is said to be a *norm* on V .

Let \mathcal{A} be a real or complex vector space. We say that \mathcal{A} is an *algebra* if \mathcal{A} is also equipped with a binary operation of multiplication

$$(8) \quad (x, y) \mapsto xy$$

for $x, y \in \mathcal{A}$ which is associative and linear in each variable x, y .

Suppose that \mathcal{A} is a real or complex algebra and that $\|\cdot\|$ is a norm on \mathcal{A} . We say that \mathcal{A} is a *normed algebra* if we also have that

$$(9) \quad \|xy\| \leq \|x\| \|y\|$$

for all $x, y \in \mathcal{A}$.

Let \mathcal{A} be a real or complex normed algebra with norm $\|\cdot\|$, and let x be an element of \mathcal{A} . For each positive integer j , let x^j be the element of \mathcal{A} which is a product of j x 's. The sequence of nonnegative real numbers $a_j = \|x^j\|$ is submultiplicative, and therefore $\{\|x^j\|^{1/j}\}_{j=1}^{\infty}$ converges as a sequence of real numbers by the earlier discussion.

Let V be a real or complex vector space. The linear transformations from V to itself form an algebra, using composition of mappings as multiplication.

Suppose further that V is equipped with a norm $\|\cdot\|$. A linear transformation T on V is said to be *bounded* if there is a nonnegative real number k such that

$$(10) \quad \|T(v)\| \leq k \|v\|$$

for all $v \in V$. In this event we can define the operator norm $\|T\|_{op}$ of T by

$$(11) \quad \|T\|_{op} = \sup\{\|T(v)\| : v \in V, \|v\| \leq 1\},$$

which is the same as the optimal value of k for the preceding inequality.

The bounded linear operators on a normed vector space also form an algebra. The operator norm defines a norm in such a way that the algebra of bounded linear operators becomes a normed algebra.

Let V be a real or complex vector space with norm $\|\cdot\|$, and observe that

$$(12) \quad d(v, w) = \|v - w\|$$

defines a metric on V . If V is complete with respect to this metric, in the sense that every Cauchy sequence in V converges to some element of V , then V is called a *Banach space*.

In any normed vector space $(V, \|\cdot\|)$, a series $\sum_{j=1}^{\infty} v_j$ with terms in V is said to converge if the sequence of partial sums converges. The series $\sum_{j=1}^{\infty} v_j$ converges absolutely if $\sum_{j=1}^{\infty} \|v_j\|$ converges as a series of nonnegative real numbers. Absolute convergence implies that the sequence of partial sums forms a Cauchy sequence.

Thus in a Banach space every absolutely convergent infinite series converges. Conversely one can show that if every absolutely convergent series in a normed vector space converges, then the space is complete.

From now on in these notes we assume that a real or complex algebra \mathcal{A} is equipped with a nonzero multiplicative identity element e . For linear operators on a vector space V , the identity transformation I on V plays this role, since the composition of any linear transformation T on V with the identity transformation I is equal to T .

We also assume that in a normed algebra \mathcal{A} the norm of the multiplicative identity element e is equal to 1. If V is a normed vector space of positive dimension, then the algebra of bounded linear operators on V equipped with the operator norm has this property, since the operator norm of the identity operator is automatically equal to 1.

By a *Banach algebra* we mean a normed algebra which is a Banach space as a normed vector space. As in the preceding paragraphs we assume that the algebra contains a nonzero multiplicative identity element whose norm is equal to 1. If V is a Banach space, then the algebra of bounded linear operators on V is a Banach algebra with respect to the operator norm.

Let \mathcal{A} be a real or complex Banach algebra with multiplicative identity element e and norm $\|\cdot\|$. For each nonnegative integer n we have that

$$(13) \quad (e - x) \left(\sum_{j=0}^n x^j \right) = \left(\sum_{j=0}^n x^j \right) (e - x) = e - x^{n+1},$$

where x^j is interpreted as being equal to e when $j = 0$.

Suppose that $\sum_{j=0}^{\infty} x^j$ converges in \mathcal{A} . As usual, this implies that

$$(14) \quad \lim_{m \rightarrow \infty} x^m = 0$$

in \mathcal{A} . It follows that $e - x$ is invertible in \mathcal{A} in this case, and that its inverse is equal to $\sum_{j=0}^{\infty} x^j$.

If $\|x\| < 1$, or if $\|x^k\| < 1$ for any positive integer k , then $\sum_{j=0}^{\infty} x^j$ converges absolutely, and hence converges. It was noted in class that convergence of $\sum_{j=0}^{\infty} x^j$ implies that $\|x^k\| < 1$ for all sufficiently large k , since $x^k \rightarrow 0$ as $k \rightarrow \infty$.

Suppose that x, y are elements of \mathcal{A} with x invertible and

$$(15) \quad \|y - x\| < \frac{1}{\|x^{-1}\|}.$$

We can write y as $x(e + x^{-1}(y - x))$ or as $(e + (y - x)x^{-1})x$ and conclude that y is invertible. Thus the invertible elements of \mathcal{A} form an open subset of \mathcal{A} .

If $x \in \mathcal{A}$, let $\rho(x)$ denote the resolvent set associated to x , which is the set of real or complex numbers λ , as appropriate, such that $\lambda e - x$ is invertible in \mathcal{A} . Let $\sigma(x)$ denote the spectrum of x , which is the set of real or complex numbers λ , as appropriate, such that $\lambda e - x$ is not invertible in \mathcal{A} .

Let x be an element of \mathcal{A} , and let λ be a real or complex number, as appropriate. If $\lambda^n e - x^n$ is invertible in \mathcal{A} , then so is $\lambda e - x$, which is the same as saying that $\lambda^n \in \rho(x^n)$ implies that $\lambda \in \rho(x)$. Hence $\lambda \in \sigma(x)$ implies that $\lambda^n \in \sigma(x^n)$.

If $|\lambda| > \|x\|$, then $\lambda e - x$ is invertible, and $\lambda \in \rho(x)$. For each positive integer n , if $|\lambda|^n > \|x^n\|$, then $\lambda \in \rho(x)$.

It follows from the earlier discussion that the resolvent set is always an open set of real or complex numbers, as appropriate. The spectrum is therefore a closed set, which is in fact compact, because it is bounded.

In the complex case a famous result states that the spectrum of an element x of \mathcal{A} is always nonempty. For if $\sigma(x)$ were empty, so that $\lambda e - x$ is

invertible for all complex numbers λ , then $(\lambda e - x)^{-1}$ would define a nonzero holomorphic \mathcal{A} -valued function on the complex plane which tends to 0 as $\lambda \rightarrow 0$. An extension of Liouville's theorem would imply that this function is equal to 0 for all λ , a contradiction.

In either the real or complex case, if $x \in \mathcal{A}$ and $\sigma(x) \neq \emptyset$, let $r(x)$ denote the spectral radius of x , which is the maximum of $|\lambda|$ over all $\lambda \in \sigma(x)$. Thus $r(x) \leq \|x^n\|^{1/n}$ for all positive integers n .

Let $R(x)$ be equal to the infimum of $\|x^n\|^{1/n}$ over all positive integers n , which is the same as the limit of $\|x^n\|^{1/n}$ as $n \rightarrow \infty$, as we have seen. In the complex case a famous result states that $r(x) = R(x)$. The idea is that $(e - \alpha x)^{-1}$ defines a holomorphic \mathcal{A} -valued function of α on the disk $|\alpha| < 1/r(x)$, and that the series $\sum_{j=0}^{\infty} \alpha^j x^j$ consequently converges everywhere on this disk.

Suppose that ϕ is a nonzero homomorphism from \mathcal{A} into the real or complex numbers, as appropriate. This means that ϕ is a linear mapping from \mathcal{A} into the real or complex numbers, and that $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in \mathcal{A}$.

In particular, $\phi(e) = 1$. More generally, if x is an invertible element of \mathcal{A} , then $\phi(x) \neq 0$. If $\phi(x) = 0$ for some $x \in \mathcal{A}$, then x is not invertible.

If $x \in \mathcal{A}$, then $\phi(x) \in \sigma(x)$. This is because ϕ applied to $\phi(x)e - x$ is equal to 0.

As a result,

$$(16) \quad |\phi(x)| \leq R(x) \leq \|x\|$$

for all $x \in \mathcal{A}$. This implies that ϕ is continuous as a mapping from \mathcal{A} into the real or complex numbers.

In the complex case for a commutative Banach algebra there are abstract arguments to the effect that every complex number in the spectrum of an element of the Banach algebra occurs as the value of a nonzero homomorphism from the algebra into the complex numbers at the element of the Banach algebra. This implies that $R(x)$ is a seminorm on \mathcal{A} . Let us check this directly for commutative normed algebras in general.

Of course $R(x)$ is always a nonnegative real number, and it is easy to see that $R(\alpha x) = |\alpha| R(x)$ for all real or complex numbers α , as appropriate, and all $x \in \mathcal{A}$. To show that $R(x)$ defines a seminorm on \mathcal{A} , it remains to check the triangle inequality.

Let $\{a_j\}_{j=0}^{\infty}$, $\{b_k\}_{k=0}^{\infty}$ be two submultiplicative sequences of nonnegative

real numbers, with $a_0 = b_0 = 1$. Define c_n for each nonnegative integer n by

$$(17) \quad c_n = \sum_{j=0}^n \binom{n}{j} a_j b_{n-j},$$

so that $c_0 = 1$.

Actually, $\{c_n\}_{n=0}^\infty$ is a submultiplicative sequence of nonnegative real numbers in this situation. Explicitly,

$$(18) \quad \sum_{j=0}^{m+n} \binom{m+n}{j} a_j b_{m+n-j} \leq \left(\sum_{k=0}^m \binom{m}{k} a_k b_{m-k} \right) \cdot \left(\sum_{l=0}^n \binom{n}{l} a_l b_{n-l} \right).$$

This is not difficult to verify, multiplying out the product on the right and using identities from the binomial theorem.

Thus $\lim_{n \rightarrow \infty} (c_n)^{1/n}$ exists. The next step is that

$$(19) \quad \lim_{n \rightarrow \infty} (c_n)^{1/n} \leq \lim_{j \rightarrow \infty} (a_j)^{1/j} + \lim_{k \rightarrow \infty} (b_k)^{1/k},$$

which is not too difficult to show.

In connection with this, notice that if t_1, \dots, t_m are nonnegative real numbers, then

$$(20) \quad \max(t_1, \dots, t_m) \leq t_1 + \dots + t_m \leq m \cdot \max(t_1, \dots, t_m).$$

If $\{t_{p,n}\}_{n=0}^\infty$, $p = 1, \dots, m$, are sequences of nonnegative real numbers, then

$$(21) \quad \limsup_{n \rightarrow \infty} \max(t_{1,n}, \dots, t_{m,n})^{1/n} = \limsup_{n \rightarrow \infty} (t_{1,n} + \dots + t_{m,n})^{1/n}.$$

Suppose that $x, y \in \mathcal{A}$, and put $a_j = \|x^j\|$, $b_k = \|y^k\|$. If c_n is as before, then

$$(22) \quad \|(x + y)^n\| \leq c_n.$$

It follows that

$$(23) \quad R(x + y) \leq R(x) + R(y),$$

which is what we wanted.

One can also check that $R(x)$ is compatible with multiplication in the same way as for a normed algebra. Namely $R(e) = 1$ and $R(xy) \leq R(x)R(y)$ when $x, y \in \mathcal{A}$ and \mathcal{A} is a commutative normed algebra.

Let us look at some special cases. Let $C(\mathbf{T})$ denote the algebra of continuous complex-valued functions on the unit circle \mathbf{T} in the complex plane,

equipped with the supremum norm, $\|f\| = \sup\{|f(z)| : z \in \mathbf{T}\}$. In this case $\|f^n\| = \|f\|^n$ for every positive integer n .

Now consider the continuous functions $f(z)$ on the unit circle \mathbf{T} which can be expressed as $\sum_{j=0}^{\infty} a_j z^j + \sum_{l=1}^{\infty} a_{-l} \bar{z}^l$, where \bar{z} denotes the complex conjugate of the complex number z and where the coefficients a_j are complex numbers such that $\sum_{j=-\infty}^{\infty} |a_j|$ converges. We define a norm $\|f\|_1$ to be equal to this sum.

This expansion for f is simply the Fourier series of f , and the point is that we restrict our attention to continuous functions whose Fourier series is absolutely summable. We certainly have that $\|f\| \leq \|f\|_1$ for such a function, which is to say that the supremum of $|f(z)|$ is bounded by $\sum_{j=-\infty}^{\infty} |a_j|$. One can check that if f_1, f_2 are two functions on the unit circle with absolutely summable Fourier series, then so is the product $f_1 f_2$ and $\|f_1 f_2\|_1 \leq \|f_1\|_1 \|f_2\|_1$.

If f is a function on the unit circle with absolutely summable Fourier series, then a famous result states that

$$(24) \quad \lim_{n \rightarrow \infty} (\|f^n\|_1)^{1/n} = \|f\|.$$

Because $R(f) = \lim_{n \rightarrow \infty} (\|f^n\|_1)^{1/n}$ defines a seminorm, it is sufficient to check that $R(f) = \|f\|$ for a dense class of functions such as trigonometric polynomials.

Now let V be the vector space of real or complex sequences $x = \{x_j\}_{j=1}^{\infty}$, as one might prefer, in which at most finitely many terms are nonzero. If p is a real number, $1 \leq p < \infty$, then we can define a norm $\|x\|_p$ on V by

$$(25) \quad \|x\|_p = \left(\sum_{j=1}^{\infty} |x_j|^p \right)^{1/p}.$$

For $p = \infty$ put

$$(26) \quad \|x\|_{\infty} = \max\{|x_j| : j \geq 1\}.$$

Let $\{\alpha_j\}_{j=1}^{\infty}$ be a monotone decreasing sequence of nonnegative real numbers. Define a linear transformation T on V by saying that the j th term of the sequence $T(x)$ is equal to $\alpha_j x_{j+1}$.

Define a sequence $\{a_l\}_{l=1}^{\infty}$ of nonnegative real numbers by

$$(27) \quad a_l = \prod_{j=1}^l \alpha_j.$$

For each positive integer l and each p , $1 \leq p \leq \infty$, a_l is the operator norm of T^l with respect to the norm $\|x\|_p$ on V . One can check directly that $\{a_l\}_{l=1}^\infty$ is a submultiplicative sequence in this case, and that $\lim_{l \rightarrow \infty} (a_l)^{1/l}$ is equal to $\lim_{j \rightarrow \infty} \alpha_j$, which is the same as the infimum of the α_j 's.

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